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B.Sc. Sem - II

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Taylor's Series :-

Statement :- If $F(x+h)$ be a given function of h, which can be expanded into a convergent series of positive ascending integral power of h then.

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{\lfloor 2} F''(x) + \cdots + \frac{h^n}{\lfloor n} F^n(x) + \cdots$$

Proof :- Let $F(x+h)$ be a function of h. which can be expanded into a convergent series of positive ascending power of h then.

$$F(x+h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \cdots$$

$$F'(x) = 0 + a_1 + 2a_2 h + 3a_3 h^2 + \cdots$$

$$a_1 = F'(x)$$

$$a_2 = \frac{F''(x)}{2}$$

This is known as Taylor's Series

Putting $x = a$ and $h = x - a$ in the above series. we get Taylor's series in power of $(x-a)$ as

$$F(x) = F(a) + \frac{(x-a)}{\lfloor 1} F'(a) + \frac{(x-a)^2}{\lfloor 2} F''(a) + \frac{(x-a)}{\lfloor n} \cdot F^n(a)$$

Also Taylor's Series at the point $x = a$.

Maclaurin's Series :-

Statement :- If $F(x)$ be a given function of x. which can be expanded into positive ascending power h then

$$F(x) = F(0) + xF'(0) + \frac{x^2}{\lfloor 2} F'(0) + \cdots$$

or If we put $x = 0$ & $h = x$ in Taylor's series

We get Macleurin's Series as

$$F(x) = F(0) + x F'(0) + \frac{x^2}{\underline{|2}} F''(0) + \cdots$$

Que:- Use Taylor's Series to P.T

$$\tan^{-1}(x+h) = \tan^{-1}x + (\tan z)\frac{\sin z}{1} - (h\sin z)^2 \frac{\sin 2z}{2}$$
$$+ (h\sin z)^3 \frac{\sin 3z}{3} \cdots$$

where $z = \cot^{-1}x$.

Proof:- Given

$$z = \cot^{-1}x \Rightarrow \cot z = x$$

$$\Rightarrow -\operatorname{cosec}^2 z \frac{dz}{dx} = 1 \quad \text{or} \quad \frac{dz}{dx} = -\sin^2 z$$

Now, let $F(x+h) = \tan^{-1}(x+h) \Rightarrow F(x) = \tan^{-1}x$

$$F'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \frac{1}{\operatorname{cosec} z}$$
$$= \sin^2 z$$

$$F''(x) = 2\sin z \cdot \cos z \cdot \frac{dz}{dx}$$
$$= \sin 2z \cdot (-\sin^2 z)$$

$$F''(x) = -\left[\sin 2z \cdot (2\sin z \cdot \cos z) \cdot \frac{dz}{dx} + \sin^2 z \cdot \cos 2z \cdot 2 \cdot \frac{dz}{dx}\right]$$

$$= -\left[(\sin 2z)^2 \cdot \sin^2 z + \sin^2 z \cdot 2\cos z\right]$$
$$= -\left[\ldots + 2\cos^2 z \cdot \sin^2 z\right]$$
$$= \sin^2 z \left[\sin 2z \cdot 2\sin z \cdot \cos z + 2\cos^2 z \cdot \sin^2 z\right]$$
$$= 2\sin z \cdot \sin^2 z \left[\sin 2z \cdot \cos z + \cos 2z \cdot \sin z\right]$$
$$= 2\sin^3 z \cdot \sin 3z$$

By Taylor Series

$$F(x+h) = F(x) + \frac{h}{\underline{|1}} F'(x) + \frac{h^2}{\underline{|2}} F'(x)$$
$$+ h^3 F''(x) + \cdots$$

$$\Rightarrow f(x+h) = \tan^{-1}x - \left(\frac{h\sin z}{1}\right)\sin z + (h\sin z)^2 \cdot \frac{\sin 2z}{2}$$

Prooved